## Independence-dimension inequality

Each vector in a linearly independent collection of n-vectors can have at most n elements. Any collection of n+1 or more n-vectors is linearly dependent.

## Basis

A collection of n linearly independent vectors.

## Orthonormal

A collection of vectors is orthonormal if it is orthogonal and $\|a_i\|=1$ for $i=1,\dots,k$. A vector of norm one is normalized. To normalize a vector, divide the vector by its norm.

$$a_i^T a_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Also, matrix M is orthonormal if $M^TM=I$.

Orthonormal vectors are linearly independent.

## Gram Schmidt

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$$

$$e_3 = \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|}$$

if numerators are nonzero, continue, otherwise terminate process. If process completes, original vectors are linearly independent

## Properties of Inner Products

**Linearity** $\langle au+bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$

**Symmetry** $\langle u,v\rangle = \langle v,u\rangle$

**Positive Definiteness** $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u=0$

$\langle cx,y\rangle = \begin{bmatrix}2\\1\end{bmatrix}^T(3cx+3y)$

$c\langle x,y\rangle = \begin{bmatrix}2\\1\end{bmatrix}^T(3cx+3cy)$

$\langle cx,y\rangle \neq c\langle x,y\rangle$

Symmetry Example

## Inverse of a Matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}, \text{ where } A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$$

Inverse exists if matrix is square with linearly independent columns.

## Kronecker Delta Function

$$\delta(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0\end{cases}$$

## QR Decomposition

Q = Gram Schmidt process so an orthonormal basis

$$R = \begin{bmatrix} a_1\cdot q_1 & a_2 q_1 & \cdots & a_n q_1 \\ 0 & a_2 q_2 & \cdots & a_n q_2 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n q_n \end{bmatrix}$$

## Orthonormal Matrices

If $A^TA=I$, then A is an orthonormal matrix.

## 4 Fundamental Subspaces

$Col(A)$ $\quad Col(A^T)$

$Null(A)$ $\quad Null(A^T)$

$\dim(col(A)) = \dim(col(A^T))$

## Rotation Matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

## Rank Nullity Theorem

$Rank(A) + \dim(N(A)) = n$

$$\cos(\theta) = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}$$

$Mx = d \quad M\alpha v = 0$

$M(x+\alpha v) = d$

$\alpha = -1$

$$M\left(\begin{bmatrix}6\\6\\3\end{bmatrix} + \alpha\begin{bmatrix}2\\3\\2\end{bmatrix}\right) = \begin{bmatrix}3\\3\end{bmatrix} \qquad x = \begin{bmatrix}4\\3\\1\end{bmatrix}$$

Find a transmitter vector x such that $Mx = \begin{bmatrix}3\\3\end{bmatrix}$ where $x_1, x_2, x_3 \in \{0,1,2,3,4,5\}$

## DTFS

Calculate period (p)

$\theta(p) = 2\pi$

Solve for p.

If p is not integer, $p = LCM(p)$

p must be integer or there is no expansion

$w_0 = \frac{2\pi}{p}$: Fundamental Frequency

Rewrite equation:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Use unit circle to assign values of k. For example, $\frac{\pi}{5}$ comes first, then $\frac{4\pi}{5}$, then $-\frac{4\pi}{5}$, then $-\frac{\pi}{5}$. Also use frequency. If $w_0 = \frac{\pi}{5}$, and there are 10 terms, $\frac{4\pi}{5}$ is the 4th term.

### Analysis Equation

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ikw_0 n}$$

### Synthesis Equation

$$x(n) = \sum_{k=0}^{p-1} X_k e^{ikw_0 n}$$

Given $x(n) = \sum_{l=-\infty}^{\infty} \delta(n-lp)$

$(-1)^n = e^{i\pi n}$

$i = e^{i\frac{\pi}{2}}$

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ikw_0 n}$$

$$\text{so, } X_k = \frac{1}{p}\left(\underbrace{x(0)}_{=1} + \underbrace{x(1)e^{-ikw_0} + \dots + x(p-1)e^{-ikw_0(p-1)}}_{=0}\right)$$

$X_k = \frac{1}{p}$ , $\quad W_0 = \frac{2\pi}{p}$

$$x(n) = \frac{1}{p}\sum_{k=0}^{p-1} e^{ikw_0 n}$$

## DTFS Puzzle

$p=3 \quad w_0 = \frac{2\pi}{3}$

$X_k = \frac{1}{p}$

$X_0 = \frac{1}{p}\cdot 0 = 0$

$X_0 + X_1 + X_2 = 0$

$X_1 + X_2 = 0$

$X_1 = -X_2$

$|X_0|^2 + |X_1|^2 + |X_2|^2 = \frac{1}{2}$

$0 + |-X_2|^2 + |X_2|^2 = \frac{1}{2}$

$+ |X_1|^2 + |X_2|^2 = \frac{1}{2}$

$2X_2^2 = \frac{1}{2}$

$X_2^2 = \frac{1}{4}$

$X_2 = \frac{1}{2}$

$X_1 = -\frac{1}{2}$

$$x(n) = \sum_{k=0}^{p-1} X_k e^{ikw_0 n}$$

$$x(n) = 0 + \left(-\frac{1}{2}\cdot e^{i\frac{2\pi}{3}n}\right) + \left(\frac{1}{2}e^{i\frac{4\pi}{3}n}\right)$$

$$x(n) = \frac{1}{2}e^{i\frac{4\pi}{3}n} - \frac{1}{2}e^{i\frac{2\pi}{3}n}$$

$$= \frac{1}{2}e^{(i\pi n + i\frac{\pi}{3}n)} - \frac{1}{2}e^{(i\pi n - i\frac{\pi}{3}n)}$$

$$= \frac{1}{2}e^{i\pi n}\left(e^{i\frac{\pi}{3}n} - e^{-i\frac{\pi}{3}n}\right)$$

$$e^{i\pi n}\, i\, \frac{\left(e^{i\frac{\pi}{3}n} - e^{-i\frac{\pi}{3}n}\right)}{2i}$$

$e^{i\pi n}e^{i\frac{\pi}{2}}\sin\left(\frac{\pi}{3}n\right)$

$e^{i(\pi n + \frac{\pi}{2})}\sin\left(\frac{\pi}{3}n\right)$

$A = \pi$

$B = \frac{\pi}{2}$

$C = \frac{\pi}{3}$

## DFT

Same as DTFS

### Analysis Equation

$$X_k = \frac{1}{\sqrt{p}}\sum_{n=0}^{p-1} x(n)e^{-ikw_0 n}$$

### Synthesis Equation

$$x(n) = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} X_k e^{ikw_0 n}$$

## Parseval's Theorem for DFT

$$\langle y,y\rangle = \langle b_1v_1 + \dots b_nv_n, b_1v_1 + \dots + b_nv_n\rangle$$

$$= b_1b_1^*\langle v_1,v_1\rangle + \dots + b_nb_n^*\langle v_n,v_n\rangle$$

$$= |b_1|^2 + \dots + |b_n|^2$$

$$= \sum_{k=1}^{n}|b_k|^2$$

Only works for normalized vectors.

$$\langle y,y\rangle = \sum_{k=1}^{n} a_k^2$$

$$\text{let } x = \begin{bmatrix} x(0) \\ \cdots \\ x(p-1)\end{bmatrix} = \begin{bmatrix} a_1 \\ \cdots \\ a_p\end{bmatrix}$$

Since DFT is an orthonormal basis, expressing x in the DFT basis would give us coefficients $b_1,\dots,b_p$. By definition of DFT, the coefficients are $X_0,\dots,X_{p-1} = b_1,\dots,b_p$

$$\text{So } \sum_{k=1}^{p} x(k)^2 = \sum_{k=1}^{p}|X_k|^2$$

## Parseval's Theorem for DTFS

$$\frac{1}{p}\sum_{k=0}^{n-1}|x(n)|^2 = \sum_{k=0}^{n-1}|X_k|^2$$

Given an invertible matrix $A \in \mathbb{R}^{n \times n}$, use the definition of matrix inverse to prove that $(A^T)^{-1} = (A^{-1})^T$

$(A^T)^{-1} = (A^{-1})^T$

$AA^{-1} = I$

$A^T (A^{-1})^T = I$

$(A^T)^{-1} =$ Inverse of $A^T = (A^{-1})^T$

---

Prove that if there are two distinct solutions $x_1$ and $x_2$ to the system $Ax = b$, then there are infinitely many solutions $x$ to this system

$Ax_1 = b$

$Ax_2 = b$

$Ax_1 - Ax_2 = b - b$

$A(x_1 - x_2) = 0$

$x_1 - x_2 \in \text{Null}(A)$

Anything of the form $x = x_1 + \alpha(x_1 - x_2)$ is a solution

$$x = x_1 + \alpha(x_1 - x_2)$$
$$A(x_1 + \alpha(x_1 - x_2))$$
$$= Ax_1 + A\alpha(x_1 - x_2)$$
$$= Ax_1 + 0$$
$$= b + 0$$
$$= b,$$

So there are an infinite number of solutions.

---

$A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$. Assume $A$ is invertible and $B$ has a nontrivial nullspace. Prove that $BA$ has a nontrivial nullspace.

B has nontrivial nullspace so there exists a vector $x, x \in \mathbb{R}^n$ such that $Bx = 0$

Since A is invertible, A and $A^{-1}$ have trivial nullspaces.

let $y = A^{-1}x$.

Since $A^{-1}$ has a trivial nullspace and $x \neq 0$, $y \neq 0$.

$$BAy = BA(A^{-1}x)$$
$$= B(A^{-1}A)x$$
$$= Bx$$
$$= 0$$

Thus, $y$ is a nonzero vector in the nullspace of BA which proves that BA has a nontrivial nullspace.

## SVD Process

① Find $A^TA$

② Solve $\det(A^TA-\lambda I)=0$

③ Use bigger $\lambda$ to solve $(A^TA-\lambda I)x=0$.
Normalize $x$.
This is $v_1$ in $V$

④ Do the same for smaller $\lambda$.
Normalize $x$.
This is $v_2$ in $V$.

⑤ $\Sigma=\begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{bmatrix}$

⑥ $u_1=\frac{1}{\sigma_1}Av_1$

$\sigma_1$ is $\sqrt{\lambda_1}$

do this for $u_2$, etc to get $U$.

if $A$ is wide, do the same but for $AA^T$ to get $\lambda$s and $u_k$s. Then, use $v_k=\frac{1}{\sigma_k}A^Tu_k$

$V^T$ applies rotation
$U$ applies rotation
$\Sigma$ applies scaling

## Gram Schmidt w/ Projection Matrices

① Normalize $a_1$ to get $q_1$. $q_1=\frac{a_1}{\|a_1\|}$

② $r_2=a_2-P_1a_2$
$P_1=a_1(a_1^Ta_1)^{-1}a_1^T$
$q_2=\frac{r_2}{\|r_2\|}$

③ $r_3=a_3-P_2a_3$
$P_2=[a_1\ a_2]\left([a_1\ a_2]^T[a_1\ a_2]\right)^{-1}[a_1\ a_2]^T$
$q_3=\frac{r_3}{\|r_3\|}$

An orthonormal matrix preserves the norm of any vector.

The largest amplification factor of a matrix is given by its largest singular value.

## Least squares estimate of x

$$\hat{x}=(A^TA)^{-1}A^Tb$$

## Projection Matrices

$P$ is a projection matrix if $P^2=P$ and $P=P^T$.
Have eigenvalues all of 1 and 0.

## Spectral Theorem

A symmetric matrix, $A$, has orthonormal eigenvectors, so that its diagonalization $A=V\Lambda V^{-1}$ can be written as $A=V\Lambda V^T$.

Matrix $A^{n\times n}$ is symmetric iff there is an orthonormal basis of $R^n$ consisting of eigenvectors of $A$.

Steady state means having an eigenvalue of 1.

## Rank 1 approximation is $B=\sigma_1u_1v_1^T$

If a matrix's columns are linearly dependent, its determinant is zero.

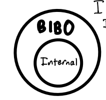
## Diagonalization

$A=V\Lambda V^{-1}$
$\det(\lambda I-A)=0$
Solve for $\lambda$s.
$\lambda$s are the eigenvalues.
Solve for $x$: $(\lambda I-A)x=0$ for all $\lambda$s. $x$ are eigenvectors which make up columns of $V$. $\lambda$ are diagonal values of $\Lambda$. For the diagonalization to be valid, the eigenvectors have to be linearly independent.

## BIBO Stability

A system is bounded input bounded output stable if it has eigenvalues with magnitude less than 1. Greater than 1 means unstable, and 1 means marginally stable.

If internally stable, also BIBO stable
If not BIBO stable, not internally stable




## PCA

① Find the mean of the data and subtract it from each data point to "demean" the data.
② Use training data to find PC1, PC2, etc., which are the $V$s from SVD.
③ Classify by projecting onto PCA Basis

$A=U\Sigma V^T$

$$\begin{bmatrix}- & a_1 & -\\ & \vdots & \\ - & a_m & -\end{bmatrix}=\begin{bmatrix}| & & |\\ u_1 & \cdots & u_r\\ | & & |\end{bmatrix}\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}\begin{bmatrix}- & v_1 & -\\ & \vdots & \\ - & v_r & -\end{bmatrix}$$

Data in rows like above $\Rightarrow$ $V$s are the PCs.
Data in columns $\Rightarrow$ $U$s are the PCs.

## Reconstruction Principle

$\tilde{A}=PC\cdot PC^T\cdot A$

## SVD

$U$ and $V$ are orthonormal

$U$: Matrix whose columns are eigenvectors of $AA^T$

$\Sigma$: Diagonal matrix of singular values (square roots of eigenvalues.

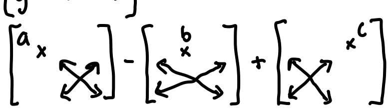
$V$: Matrix whose columns are eigenvectors of $A^TA$.

## Determinants

2x2: $ad-bc$ $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

3x3: $a(ei-fh)-b(di-fg)+c(dh-eg)$

$\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$



## Difference Equations

Order is largest delay

## Full SVD

$$\begin{array}{c c} & n\\ m & \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\end{array}=\begin{array}{c c} & m\\ m & \begin{bmatrix} \\ U \\ \\ \end{bmatrix}\end{array}\begin{array}{c c} & n\\ m & \begin{bmatrix} \\ \Sigma \\ \\ \end{bmatrix}\end{array}\begin{array}{c c} & n\\ n & \begin{bmatrix} \\ V^T \\ \\ \end{bmatrix}\end{array}$$

## Compact SVD

$$\begin{array}{c c} & n\\ m & \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\end{array}\quad\begin{array}{c c} & r\\ m & \begin{bmatrix} \\ U \\ \\ \end{bmatrix}\end{array}\begin{array}{c c} & r\\ r & \begin{bmatrix} \\ \Sigma \\ \\ \end{bmatrix}\end{array}\begin{array}{c c} & n\\ r & \begin{bmatrix} \\ V^T \\ \\ \end{bmatrix}\end{array}$$

## Outer form

$A=\sigma_1u_1v_1^T+\dots+\sigma_ru_rv_r^T$

Rank 1 Approximation is the 1st term

## Pseudoinverse Identities

① if $A^{-1}$ exists, $A^+=A^{-1}$
② $(A^+)^+=A$
③ $(A^T)^+=(A^+)^T$
④ if $\alpha\neq0$, $(\alpha A)^+=\alpha^{-1}A^+$ (scalar distributivity)
⑤ $AA^+A=A$ (weak left inverse property)
⑥ $A^+AA^+=A^+$ (weak right inverse property)
⑦ $AA^+=U_rU_r^T$ ($AA^+$ is projection onto Col(A))
⑧ $A^+A=V_rV_r^T$ ($A^+A$ is projection onto Col($A^T$))

A triangular matrix has its eigenvalues along the diagonal

$\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}\rightarrow\lambda=1,2$

## Compact Pseudoinverse

From compact SVD,
$A^+=V\Sigma^{-1}U^T$

## Error vector for Least Squares

$E=b-Ax$

## Pseudoinverse

$A^+=V\Sigma^+U^T$

$\Sigma^+=\Sigma^T$, and take the inverse of each value along the diagonal.
For diagonal matrices, $A^+=\Sigma^+$

if $A^{-1}$ exists: $A^{-1}=V\Sigma^{-1}U^T$
if $A^{-1}$ does not exist: $A^{-1}=V\Sigma^+U^T$

$Ax=b$
if $A$ is square $\Rightarrow$ 1 solution (if $A$ is invertible)
if $A$ is tall $\Rightarrow$ "overdetermined", no solutions
if $A$ is wide $\Rightarrow$ "underdetermined", many solutions

$\|Ax\|^2 = (Ax)^T(Ax) = x^TA^TAx$
$= x^T(U\Sigma V^T)^T(U\Sigma V^T)x$
$= x^T\Sigma^TV\Sigma V^Tx$

$\|V^Tx\|^2 = (V^Tx)^T(V^Tx)$
$= Vx^TV^Tx = x^Tx = \|x\|^2$
Since $\|V^Tx\|^2 = \|x\|^2 \to \|V^Tx\| = \|x\|$

## Continuous Time System

All $\sigma$'s $< 0 \Rightarrow$ Stable
Any $\sigma$'s $> 0 \Rightarrow$ Unstable
All $\sigma$'s $\leq 0 \Rightarrow$ Marginally Stable

$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$

$\lambda = \sigma + jw$
$e^{\lambda t} = e^{\sigma t}e^{jwt} = e^{\sigma t}(\cos(wt) + j\sin(wt))$

## Discrete Time System

All R's $< 1 \Rightarrow$ Stable
Any R's $> 1 \Rightarrow$ Unstable
All R's $\leq 1 \Rightarrow$ Marginally Stable

$x[i] = a^ix[0] + \sum_{k=0}^{i-1} a^{i-1-k}bu[k]$

$\lambda = Re^{j\theta}$
$\lambda^i = (Re^{j\theta})^i = R^ie^{ji\theta}$
Check magnitude of eigenvalues

## Differential Equations

$\dot{x} = x_h + x_p$

1. Find $x_h$ by ignoring anything without x term.
2. Find $x_p$ (Guess) $\to$ (constant, exponential, powers
3. Solve for constants using initial conditions.

Order is highest derivative

## Stability

A system is state space ("internally") stable iff for every i.c., and with no input, the state trajectory is bounded.

sum of diagonals = trace(A) = sum of eigenvalues
product of eigenvalues = det(A)
symmetric matrix $\Rightarrow$ orthogonal eigenvectors
$A^T$ and $A$ have same eigenvalues if
$\lambda = 0 \Rightarrow A$ is not invertible, has a null space

## Calc Rules

### Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

### Chain Rule

$$\frac{d}{dx}\left[(f(x))^n\right] = n(f(x))^{n-1} \cdot f'(x)$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

### Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Exponential

$\frac{dx}{dt} = ax \; dt \Rightarrow x = Ce^{at}$
since $\int \frac{dx}{x} = \ln x$ and $\int a\,dt = at$

### Power

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

### Constants

$$\int C_1 dx = C_1x + C_2$$

### Product Rule

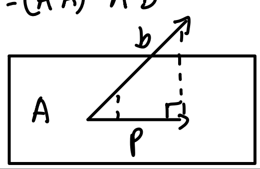
$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

$\dot{x} = \Lambda x$
$x = e^{\Lambda t}x(0)$

$A = V\Lambda V^{-1}$
$A^n = V\Lambda^nV^{-1}$

$e^{At} = Ve^{\Lambda t}V^{-1}$ | $A^i = V\Lambda^iV^{-1}$

$\hat{x} = (A^TA)^{-1}A^Tb$




$p = A\hat{x} = A(A^TA)^{-1}A^Tb$
$\hat{x}$ is the coefficients by which you multiply the basis vectors to get p.

## SVD Properties

1. $Col(U_r) = Col(A)$
2. $Col(V_r) = Col(A^T)$
3. $Col(U_{m-r}) = Null(A^T)$
4. $Col(V_{n-r}) = Null(A)$
5. $AA^T = U\Sigma\Sigma^TU^T$
6. $A^TA = V\Sigma^T\Sigma V^T$
7. $AV_r = U_r\Sigma_r$

## Pseudoinverse solves:

**Least Squares** (Overdetermined Systems)
$A^+ = (A^TA)^{-1}A^T$

**Min-Norm Solution** (Underdetermined Systems)
$A^+ = A^T(AA^T)^{-1}$

Columns are linearly dependent $\Rightarrow$ has eigenvalue of 0.

## Diagonal Matrix Eigenstuff

Eigen values are the diagonals and eigenvectors are unit vectors.

$Ax = b$ no soln
$A^TA\hat{x} = A^Tb$ "normal eqn"
$\hat{x} = (A^TA)^{-1}A^Tb$
$p = A\hat{x} = A(A^TA)^{-1}A^Tb = proj_A(B)$
$P = A(A^TA)^{-1}A^T$

$\begin{vmatrix} \alpha a & \alpha b \\ c & d \end{vmatrix} = \alpha\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$\det(A^k) = \det(A)^k$ $\quad \det(A^T) = \det(A)$
$\det(AB) = \det(A)\det(B)$

VDE equations

$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$ $\quad \Lambda^i = \begin{bmatrix} \lambda_1^i & & \\ & \ddots & \\ & & \lambda_n^i \end{bmatrix}$

$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
$x[i] = A^ix[0] + \sum_{k=0}^{i-1} A^{i-1-k}Bu[k]$